Cambridge Assessment International Education

Cambridge International AS & A Level

CANDIDATE NAME

CENTRE NUMBER

CANDIDATE NUMBER

2085422030

MATHEMATICS **9709/33**

Paper 3 Pure Mathematics 3 **October/November 2020**

1 hour 50 minutes

You must answer on the question paper.

You will need: List of formulae (MF19)

INSTRUCTIONS

- Answer **all** questions.
- Use a black or dark blue pen. You may use an HB pencil for any diagrams or graphs.
- Write your name, centre number and candidate number in the boxes at the top of the page.
- Write your answer to each question in the space provided.
- Do **not** use an erasable pen or correction fluid.
- Do **not** write on any bar codes.
- If additional space is needed, you should use the lined page at the end of this booklet; the question number or numbers must be clearly shown.
- You should use a calculator where appropriate.
- You must show all necessary working clearly; no marks will be given for unsupported answers from a calculator.
- Give non-exact numerical answers correct to 3 significant figures, or 1 decimal place for angles in degrees, unless a different level of accuracy is specified in the question.

INFORMATION

- The total mark for this paper is 75.
- The number of marks for each question or part question is shown in brackets [ ].

This document has **20** pages. Blank pages are indicated.

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1 Solve the inequality $2 - 5x > 2|x - 3|$. [4]

$$(2-5x)^2 > (2(x-3))^2$$

$$4 - 20x + 25x^2 > 4(x^2 - 6x + 9)$$

$$4 - 20x + 25x^2 > 4x^2 - 24x + 36$$

$$21x^2 + 4x - 32 > 0$$

$$x = \frac{8}{7}, -\frac{4}{3}$$

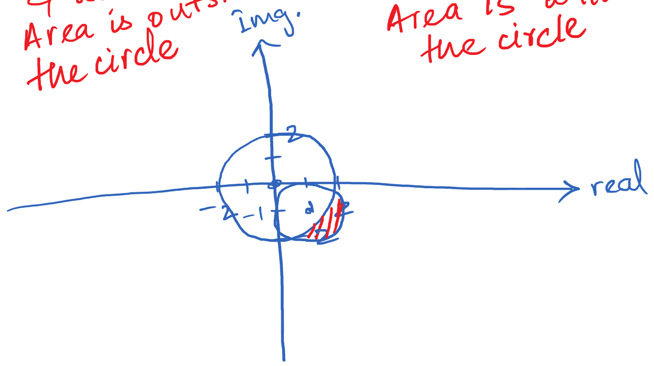
NOT VALID FOR $2-5x > 2|x-3|$

$$x < -\frac{4}{3}$$

2 On a sketch of an Argand diagram, shade the region whose points represent complex numbers z satisfying the inequalities $|z| \geqslant 2$ and $|z - 1 + \mathrm{i}| \leqslant 1$. [4]

circle with centre (0,0) & radius 2
Area is outside the circle

$z-(1-i) \leq 1$ is circle with centre (1,-1) & radius 1
Area is within the circle



3 The parametric equations of a curve are

$$x = 3 - \cos 2\theta, \quad y = 2\theta + \sin 2\theta,$$

for $0 < \theta < \frac{1}{2}\pi$.

Show that $\dfrac{dy}{dx} = \cot\theta$. [5]

$\dfrac{dx}{d\theta} = 2\sin 2\theta$ & $\dfrac{dy}{d\theta} = 2 + 2\cos 2\theta$

$\dfrac{dy}{dx} \longrightarrow \dfrac{dy}{d\theta} \times \dfrac{d\theta}{dx}$

$\dfrac{2+2\cos 2\theta}{2\sin 2\theta} \longrightarrow \dfrac{\cancel{2}(1+\cos 2\theta)}{\cancel{2}\sin 2\theta}$

$\dfrac{\cancel{2}\cos^{\cancel{2}}\theta}{\cancel{2}\sin\theta\cancel{\cos\theta}} = \dfrac{\cos\theta}{\sin\theta} \approx \cot\theta$

4 Solve the equation

$$\log_{10}(2x + 1) = 2\log_{10}(x + 1) - 1.$$

Give your answers correct to 3 decimal places. [6]

$$\log_{10}(2x+1) = \log_{10}(x+1)^2 - 1$$

$$\log_{10}(x+1)^2 - \log_{10}(2x+1) = 1$$

$$\log_{10}\left(\frac{(x+1)^2}{2x+1}\right) = 1$$

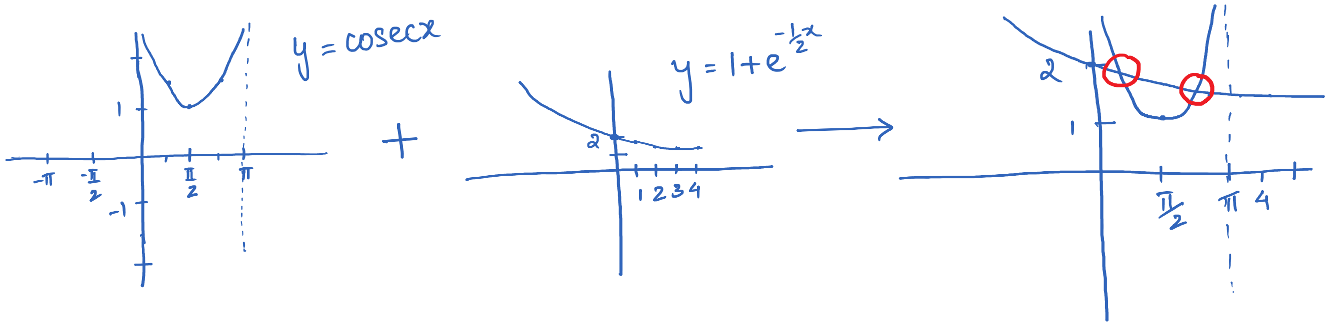
$$\frac{(x+1)^2}{2x+1} = 10^1$$

$$x^2 + 2x + 1 = 20x + 10$$

$$x^2 - 18x - 9 = 0$$

$$x = 18.487, -0.487$$

5 **(a)** By sketching a suitable pair of graphs, show that the equation $\operatorname{cosec} x = 1 + e^{-\frac{1}{2}x}$ has exactly two roots in the interval $0 < x < \pi$. [2]



Intersection of both graphs are the roots of $\operatorname{cosec} x = 1 + e^{-\frac{1}{2}x}$

(b) The sequence of values given by the iterative formula

$$x_{n+1} = \pi - \sin^{-1}\left(\frac{1}{e^{-\frac{1}{2}x_n} + 1}\right),$$

with initial value $x_1 = 2$, converges to one of these roots.

Use the formula to determine this root correct to 2 decimal places. Give the result of each iteration to 4 decimal places. [3]

$x_{1+1} = \pi - \sin^{-1}\left(\frac{1}{e^{-\frac{1}{2}\times 2} + 1}\right) \longrightarrow x_2 = 2.3217$

$x_{2+1} = \pi - \sin^{-1}\left(\frac{1}{e^{-\frac{1}{2}\times 2.3217} + 1}\right) \longrightarrow x_3 = 2.2760$

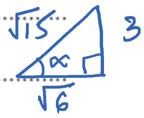
$x_4 = \pi - \sin^{-1}\left(\frac{1}{e^{-\frac{1}{2}\times 2.2760} + 1}\right) \longrightarrow 2.2824$

$x_5 = 2.2815$, $x_6 = 2.2816$, $x_7 = 2.2816$

$\therefore x = 2.28$

6 **(a)** Express $\sqrt{6}\cos\theta + 3\sin\theta$ in the form $R\cos(\theta - \alpha)$, where $R > 0$ and $0° < \alpha < 90°$. State the exact value of R and give α correct to 2 decimal places. [3]

$R\cos(\theta-\alpha) = \underbrace{R\cos\theta\cos\alpha}_{\sqrt{6}} + \underbrace{R\sin\theta\sin\alpha}_{3}$



$R\cos\alpha = \sqrt{6}$ & $R\sin\alpha = 3$

$\tan\alpha = \dfrac{3}{\sqrt{6}} \longrightarrow \alpha = 50.77°$ //

$R\sin\alpha = 3$ & $\sin\alpha = \dfrac{3}{\sqrt{15}}$

$\dfrac{3}{\sqrt{15}}R = 3$ so $R = \sqrt{15}$ //

(b) Hence solve the equation $\sqrt{6}\cos\frac{1}{3}x + 3\sin\frac{1}{3}x = 2.5$, for $0° < x < 360°$. [4]

$$\sqrt{15}\cos\left(\frac{x}{3} - 50.77\right) = 2.5$$

$$\cos\left(\frac{x}{3} - 50.77°\right) = \frac{\sqrt{15}}{6}$$

cos is +ve in 1st & 4th quadrants

$$\frac{x}{3} - 50.77° = \cos^{-1}\left(\frac{\sqrt{15}}{6}\right)$$

$$\frac{x}{3} - 50.77° = 49.8° \quad \& \quad \frac{x}{3} - 50.77° = 310.203°$$

(invalid bcz out of range)

$$x = 301.7°$$

7 (a) Verify that $-1 + \sqrt{5}\mathrm{i}$ is a root of the equation $2x^3 + x^2 + 6x - 18 = 0$. [3]

$2(-1+\sqrt{5}i)^3 + (-1+\sqrt{5}i)^2 + 6(-1+\sqrt{5}i) - 18 = 0$

$(-1+\sqrt{5}i)^3 = 14 - 2\sqrt{5}i$ & $(-1+\sqrt{5}i)^2 = -4 - 2\sqrt{5}i$

$2(14 - 2\sqrt{5}i) + (-4 - 2\sqrt{5}i) - 6 + 6\sqrt{5}i - 18 = 0$

$28 - 4\sqrt{5}i - 4 - 2\sqrt{5}i - 6 + 6\sqrt{5}i - 18 = 0$

$0 = 0$

(b) Find the other roots of this equation. [4]

Conjugate of $-1+\sqrt{5}i$ is also a root which is $-1-\sqrt{5}i$

Sum of $-1+\sqrt{5}i$ and $-1-\sqrt{5}i$ is -2

Product of $-1+\sqrt{5}i$ & $-1-\sqrt{5}i$ is 6

Quadratic factor with zeros at $x=-1+\sqrt{5}i$ and $x=-1-\sqrt{5}i$

is $x^2-x(\text{Sum of roots})+(\text{Product of roots})$

x^2+2x+6

$$
\begin{array}{r|l}
 & 2x-3 \\
\hline
x^2+2x+6 & 2x^3+x^2+6x-18 \\
 & -(2x^3+4x^2+12x) \\
\hline
 & -3x^2-6x-18 \\
 & -(-3x^2-6x-18) \\
\hline
 & \times
\end{array}
$$

The other root is $2x-3=0 \longrightarrow x=\frac{3}{2}$

8 The coordinates (x, y) of a general point of a curve satisfy the differential equation

$$x\frac{\mathrm{d}y}{\mathrm{d}x} = (1 - 2x^2)y,$$

for $x > 0$. It is given that $y = 1$ when $x = 1$.

Solve the differential equation, obtaining an expression for y in terms of x. [6]

$$\int \frac{dy}{y} = \int \frac{1-2x^2}{x}\, dx$$

$$\ln y = \int \frac{1}{x} - 2x \, dx$$

$$\ln y = \ln x - x^2 + c$$

Use $(1,1)$ to find the value of c

$$\ln 1 = \ln 1 - 1^2 + c \quad \text{so} \quad c = 1$$

$$\ln y = \ln x - x^2 + 1$$

$$\ln y - \ln x = 1 - x^2$$

$$\ln\left(\frac{y}{x}\right) = 1 - x^2$$

$$y = xe^{1-x^2}$$

9 Let $f(x) = \dfrac{8 + 5x + 12x^2}{(1 - x)(2 + 3x)^2}$.

(a) Express $f(x)$ in partial fractions. [5]

$$\frac{A}{1-x} + \frac{B}{2+3x} + \frac{C}{(2+3x)^2} \equiv 8 + 5x + 12x^2$$

$$A(2+3x)^2 + B(2+3x)(1-x) + C(1-x) \equiv 8 + 5x + 12x^2$$

$$A(4+12x+9x^2) + B(2+x-3x^2) + C(1-x) \equiv 8 + 5x + 12x^2$$

Compare the coefficients of x^2 on both sides

$$9A - 3B = 12 \longrightarrow 3A - 4 = B \quad (1)$$

Compare the coefficients of x on both sides

$$12A + B - C = 5 \quad (2)$$

Compare the constants on both sides

$$4A + 2B + C = 8 \quad (3)$$

(1) into (2)

$$12A + (3A - 4) - C = 5 \longrightarrow 15A - C = 9 \quad (4)$$

(1) into (3)

$$4A + 2(3A - 4) + C = 8 \longrightarrow 10A + C = 16 \quad (5)$$

$$(4) = (5)$$

$$15A - 9 = 16 - 10A$$

$$25A = 25 \text{ so } A = 1, \ B = -1 \ \& \ C = 6$$

$$\frac{1}{1-x} - \frac{1}{2+3x} + \frac{6}{(2+3x)^2}$$

(b) Hence obtain the expansion of $f(x)$ in ascending powers of x, up to and including the term in x^2. [5]

$(1-x)^{-1} - 1(2+3x)^{-1} + 6(2+3x)^{-2}$

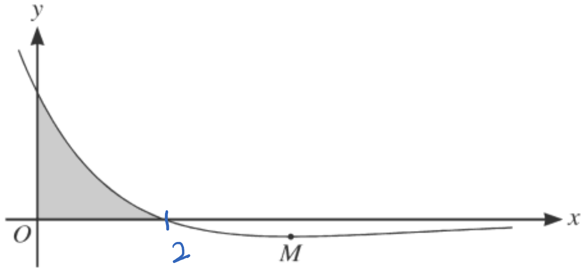
$(2+3x)^{-1} \longrightarrow 2^{-1}\left(1+\frac{3}{2}x\right)^{-1} \longrightarrow \frac{1}{2}\left(1+\frac{3}{2}x\right)^{-1}$

$(2+3x)^{-2} \longrightarrow 2^{-2}\left(1+\frac{3}{2}x\right)^{-2} \longrightarrow \frac{1}{4}\left(1+\frac{3}{2}x\right)^{-2}$

$(1+x+x^2) - \frac{1}{2}\left(1-\frac{3}{2}x+\frac{9}{4}x^2\right) + \frac{6}{4}\left(1-3x+\frac{27}{4}x^2\right)$

$2 - \frac{11}{4}x + 10x^2$

10



The diagram shows the curve $y = (2-x)e^{-\frac{1}{2}x}$, and its minimum point M.

(a) Find the exact coordinates of M. [5]

$\frac{dy}{dx}$ of $(2-x)e^{-\frac{1}{2}x}$:

$u = 2-x$ & $v = e^{-\frac{1}{2}x}$ so $\frac{du}{dx} = -1$ &

$\frac{dv}{dx} = \frac{-e^{-\frac{1}{2}x}}{2}$

$\frac{dy}{dx} = 0 \rightarrow u\frac{dv}{dx} + v\frac{du}{dx} = 0$

$(2-x)\left(\frac{-e^{-\frac{1}{2}x}}{2}\right) + e^{-\frac{1}{2}x}(-1) = 0$

$-e^{-\frac{1}{2}x} + \frac{xe^{-\frac{1}{2}x}}{2} - e^{-\frac{1}{2}x} = 0$

$\frac{x\cancel{e^{-\frac{1}{2}x}}}{2} = 2\cancel{e^{-\frac{1}{2}x}} \rightarrow x = 4$ & $y = \frac{-2}{e^2}$

(b) Find the area of the shaded region bounded by the curve and the axes. Give your answer in terms of e. [5]

$$\int_0^2 (2-x)e^{-\frac{1}{2}x}$$

$$\int_0^2 2e^{-\frac{1}{2}x} - \int_0^2 xe^{-\frac{1}{2}x}$$

$$-4e^{-\frac{1}{2}x} - \int xe^{-\frac{1}{2}x}$$

$u = x$ & $\frac{dv}{dx} = e^{-\frac{1}{2}x}$

$\frac{du}{dx} = 1$ $\quad v = -2e^{-\frac{1}{2}x}$

$$\int xe^{-\frac{1}{2}x} \longrightarrow uv - \int v\frac{du}{dx}$$

$$\left[-2xe^{-\frac{1}{2}x}\right] - \int -2e^{-\frac{1}{2}x} \times 1$$

$$\int xe^{-\frac{1}{2}x} \longrightarrow -2xe^{-\frac{1}{2}x} + 2\left[-2e^{-\frac{1}{2}x}\right]$$

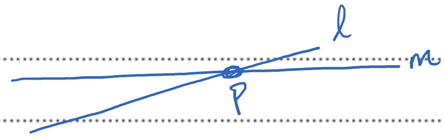
$$\int xe^{-\frac{1}{2}x} \longrightarrow -2xe^{-\frac{1}{2}x} - 4e^{-\frac{1}{2}x}$$

$$-4e^{-\frac{1}{2}x} - \int xe^{-\frac{1}{2}x} \longrightarrow -4e^{-\frac{1}{2}x} + 2xe^{-\frac{1}{2}x} + 4e^{-\frac{1}{2}x}$$

$$\left[2xe^{-\frac{1}{2}x}\right]_0^2 = \frac{4}{e}$$

11 Two lines have equations $\mathbf{r} = \mathbf{i} + 2\mathbf{j} + \mathbf{k} + \lambda(a\mathbf{i} + 2\mathbf{j} - \mathbf{k})$ and $\mathbf{r} = 2\mathbf{i} + \mathbf{j} - \mathbf{k} + \mu(2\mathbf{i} - \mathbf{j} + \mathbf{k})$, where a is a constant.

(a) Given that the two lines intersect, find the value of a and the position vector of the point of intersection. [5]



Point P on l has position vector $\begin{pmatrix} 1+a\lambda \\ 2+2\lambda \\ 1-\lambda \end{pmatrix}$ &

that on m has position vector $\begin{pmatrix} 2+2\mu \\ 1-\mu \\ -1+\mu \end{pmatrix}$

$1 + a\lambda = 2 + 2\mu$

$2 + 2\lambda = 1 - \mu$ so $\mu = -2\lambda - 1$ (1)

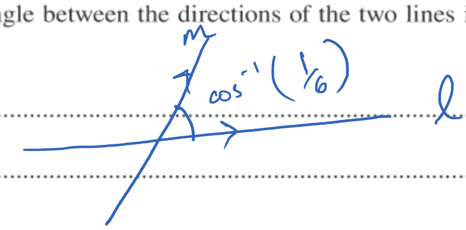
$1 - \lambda = -1 + \mu$ so $\mu = 2 - \lambda$ (2)

$-2\lambda - 1 = 2 - \lambda$ so $\lambda = -3$ & $\mu = 5$

$1 - 3a = 2 + 10$ so $a = -11/3$

Point P is $12\mathbf{i} - 4\mathbf{j} + 4\mathbf{k}$

(b) Given instead that the acute angle between the directions of the two lines is $\cos^{-1}\left(\frac{1}{6}\right)$, find the two possible values of a. [6]



d_1 of m is $ai+2j-k$ & d_2 of m is $2i-j+k$

$$\begin{pmatrix} a \\ 2 \\ -1 \end{pmatrix} \cdot \begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix} = \sqrt{5+a^2}\sqrt{6}\cos\theta$$

$$\frac{2a-3}{1/6\sqrt{6}} = \sqrt{5+a^2} \rightarrow \frac{4a^2-12a+9}{1/6} = 5+a^2$$

$$4a^2-12a+9 = \frac{5}{6}+\frac{a^2}{6}$$

$$\frac{23}{6}a^2-12a+\frac{49}{6} = 0$$

$$a = \frac{49}{23}, 1$$

Additional Page

If you use the following lined page to complete the answer(s) to any question(s), the question number(s) must be clearly shown.