Cambridge Assessment International Education

# Cambridge International AS & A Level

CANDIDATE NAME

CENTRE NUMBER

CANDIDATE NUMBER

*1862605169*

**MATHEMATICS** **9709/33**

Paper 3 Pure Mathematics 3 **May/June 2020**

**1 hour 50 minutes**

You must answer on the question paper.

You will need: List of formulae (MF19)

**INSTRUCTIONS**

- Answer **all** questions.
- Use a black or dark blue pen. You may use an HB pencil for any diagrams or graphs.
- Write your name, centre number and candidate number in the boxes at the top of the page.
- Write your answer to each question in the space provided.
- Do **not** use an erasable pen or correction fluid.
- Do **not** write on any bar codes.
- If additional space is needed, you should use the lined page at the end of this booklet; the question number or numbers must be clearly shown.
- You should use a calculator where appropriate.
- You must show all necessary working clearly; no marks will be given for unsupported answers from a calculator.
- Give non-exact numerical answers correct to 3 significant figures, or 1 decimal place for angles in degrees, unless a different level of accuracy is specified in the question.

**INFORMATION**

- The total mark for this paper is 75.
- The number of marks for each question or part question is shown in brackets [ ].

This document has **20** pages. Blank pages are indicated.

JC20 06_9709_33/RP

1 Solve the inequality $|2x - 1| > 3|x + 2|$. [4]

$(2x-1)^2 > (3(x+2))^2$

$4x^2 - 4x + 1 > 9(x^2 + 4x + 4)$

$4x^2 - 4x + 1 > 9x^2 + 36x + 36$

$5x^2 + 40x + 35 < 0$

$x^2 + 8x + 7 < 0$

$x = -1, -7$

$-7 < x < -1$

**2** Find the exact value of $\int_0^1 (2-x)e^{-2x}\,dx$. [5]

$$\int_0^1 2e^{-2x} - \int_0^1 xe^{-2x}$$

$$2\left[\frac{e^{-2x}}{-2}\right] - \int xe^{-2x} \longrightarrow \text{divide them to } u = x \text{ \& } \frac{dv}{dx} = e^{-2x}$$

$$\int xe^{-2x} \rightarrow uv - \int v\frac{du}{dx} \qquad \text{so } \frac{du}{dx} = 1 \text{ \& } v = \frac{-e^{-2x}}{2}$$

$$\left[\frac{-xe^{-2x}}{2}\right] - \int \frac{-e^{-2x}}{2} \longrightarrow \frac{-xe^{-2x}}{2} + \frac{1}{2}\int e^{-2x}$$

$$\frac{-xe^{-2x}}{2} + \frac{1}{2}\left[\frac{e^{-2x}}{-2}\right] \longrightarrow \frac{-xe^{-2x}}{2} - \frac{e^{-2x}}{4}$$

$$\int_0^1 2e^{-2x} - \int_0^1 xe^{-2x} = -e^{-2x} - \left(\frac{-xe^{-2x}}{2} - \frac{e^{-2x}}{4}\right)$$

$$= \left[\frac{-3}{4}e^{-2x} + \frac{xe^{-2x}}{2}\right]_0^1$$

$$= \left[\frac{-3e^{-2}}{4} + \frac{e^{-2}}{2}\right] - \left[-\frac{3}{4}\right] = \frac{3-e^{-2}}{4}$$

**3** **(a)** Show that the equation

$$\ln(1 + e^{-x}) + 2x = 0$$

can be expressed as a quadratic equation in $e^x$. [2]

$\ln(1+e^{-x}) = -2x$

$1+e^{-x} = e^{-2x}$

$e^{-2x} - e^{-x} - 1 = 0$

Suppose $u = e^{-x}$

$u^2 - u - 1 = 0$

**(b)** Hence solve the equation $\ln(1 + e^{-x}) + 2x = 0$, giving your answer correct to 3 decimal places. [4]

$u^2 + u - 1 = 0$

$u = \dfrac{-1 \pm \sqrt{5}}{2}$

$e^{-x} = \dfrac{-1+\sqrt{5}}{2}$ & $e^{-x} = \dfrac{-1-\sqrt{5}}{2}$

$-x \ln e = \ln\left(\dfrac{-1+\sqrt{5}}{2}\right)$

$x = -0.481$

**4** The equation of a curve is $y = x\tan^{-1}(\frac{1}{2}x)$. $x$ & $\tan^{-1}\left(\frac{x}{2}\right)$

**(a)** Find $\frac{dy}{dx}$. [3]

$u = x$ & $v = \tan^{-1}\left(\frac{x}{2}\right)$

$\frac{du}{dx} = 1$ & $\frac{dv}{dx} = \frac{2}{4+x^2}$

$\frac{dy}{dx} = u\frac{dv}{dx} + v\frac{du}{dx}$

$= x\left(\frac{2}{4+x^2}\right) + \tan^{-1}\left(\frac{x}{2}\right)$

$= \frac{2x}{4+x^2} + \tan^{-1}\left(\frac{x}{2}\right)$

Differentiation of $\tan^{-1}(x)$ is $\frac{1}{1+x^2}$

so $\tan^{-1}\left(\frac{x}{2}\right) = \frac{1}{1+\left(\frac{x}{2}\right)^2} \times \frac{1}{2}$

$= \frac{\frac{1}{2}}{1+\frac{x^2}{4}}$

To cancel $x^2$ fraction multiply with 4 the whole fraction

$= \frac{2}{4+x^2}$

**(b)** The tangent to the curve at the point where $x = 2$ ($y = \frac{\pi}{2}$) meets the $y$-axis at the point with coordinates $(0, p)$.

Find $p$. [3]

Equation of tangent to curve at $x=2 \rightarrow$ ?

Gradient of curve at $x=2$ is equal to gradient of tangent.

For Gradient of curve at $x=2$, use $\frac{dy}{dx}$ & substitute 2

Gradient of curve $\rightarrow \frac{2(2)}{4+2^2} + \tan^{-1}\left(\frac{2}{2}\right) = \frac{\pi}{4} + \frac{1}{2} = \frac{\pi+2}{4}$

Eq of tangent $\rightarrow \frac{\pi}{2} = \frac{\pi+2}{4}(2) + c \rightarrow c = \frac{\pi}{2} - \frac{(\pi+2)}{2} \rightarrow c = -1$

y intercept of tangent

hence $p = -1$

**5** By first expressing the equation

$$\tan\theta\tan(\theta+45°) = 2\cot 2\theta$$

as a quadratic equation in $\tan\theta$, solve the equation for $0° < \theta < 90°$. [6]

$\tan(A+B) = \dfrac{\tan A + \tan B}{1-\tan A\tan B}$ $\cot 2\theta = \dfrac{1}{\tan 2\theta}$ & $\tan 2\theta = \dfrac{2\tan\theta}{1-\tan^2\theta}$

$\tan\theta\left(\dfrac{\tan\theta+\tan 45}{1-\tan 45\tan\theta}\right) = \dfrac{2-2\tan^2\theta}{2\tan\theta}$ $\cot 2\theta = \dfrac{1-\tan^2\theta}{2\tan\theta}$

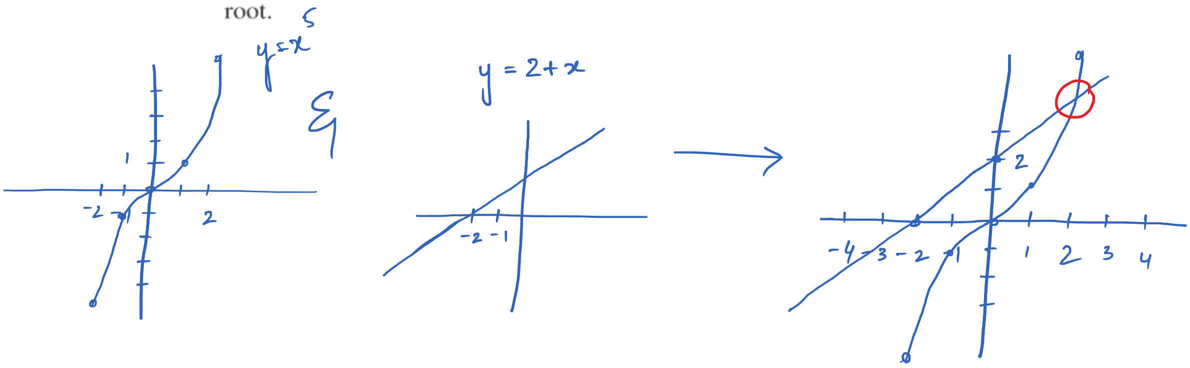
$\dfrac{\tan^2\theta+\tan\theta}{1-\tan\theta} = \dfrac{1-\tan^2\theta}{\tan\theta}$

$\dfrac{\tan\theta(\cancel{\tan\theta+1})}{1-\tan\theta} = \dfrac{(\cancel{1+\tan\theta})(1-\tan\theta)}{\tan\theta}$

$\tan^2\theta = (1-\tan\theta)^2$

$\tan\theta = \dfrac{1}{2} \longrightarrow \theta = 26.6°$

6 **(a)** By sketching a suitable pair of graphs, show that the equation $x^5 = 2 + x$ has exactly one real root. [2]




**(b)** Show that if a sequence of values given by the iterative formula

$$x_{n+1} = \frac{4x_n^5 + 2}{5x_n^4 - 1}$$

converges, then it converges to the root of the equation in part **(a)**. [2]

$$x = \frac{4x^5 + 2}{5x^4 - 1}$$

$$5x^5 - x = 4x^5 + 2$$

$$x^5 = 2 + x$$

(c) Use the iterative formula with initial value $x_1 = 1.5$ to calculate the root correct to 3 decimal places. Give the result of each iteration to 5 decimal places. [3]

$x_1 = 1.5$

$x_2 = 1.33162$

$x_3 = 1.27352$

$x_4 = 1.26724$

$x_5 = 1.26717$

$x_6 = 1.26717$

$\therefore x = 1.267$

**7** Let $f(x) = \dfrac{2}{(2x-1)(2x+1)}$.

**(a)** Express f(x) in partial fractions. [2]

$$\frac{A}{2x-1} + \frac{B}{2x+1} = 2$$

$$A(2x+1) + B(2x-1) = 2$$

$$2Ax + A + 2Bx - B = 2$$

$$A - B = 2 \quad \text{so} \quad A = 2 + B$$

$$2A + 2B = 0 \quad \text{so} \quad 2(2+B) + 2B = 0$$

$$4 + 4B = 0 \quad \text{so} \quad B = -1 \ \& \ A = 1$$

$$\frac{1}{2x-1} - \frac{1}{2x+1}$$

**(b)** Using your answer to part **(a)**, show that

$$(f(x))^2 = \frac{1}{(2x-1)^2} - \frac{1}{2x-1} + \frac{1}{2x+1} + \frac{1}{(2x+1)^2}.$$ [2]

$$\left(\frac{1}{2x-1} - \frac{1}{2x+1}\right)^2 = (a-b)^2 \longrightarrow a^2 - 2ab + b^2$$

$$\frac{1}{(2x-1)^2} - 2\left(\frac{1}{2x-1} \times \frac{1}{2x+1}\right) + \frac{1}{(2x+1)^2}$$

$$\frac{1}{(2x-1)^2} - \frac{2}{(2x-1)(2x+1)} + \frac{1}{(2x+1)^2}$$

Its expansion from part (a)

$$\frac{1}{(2x-1)^2} - \frac{1}{2x-1} + \frac{1}{2x+1} + \frac{1}{(2x+1)^2}$$

**(c)** Hence show that $\int_1^2 (\mathrm{f}(x))^2\,\mathrm{d}x = \frac{2}{5} + \frac{1}{2}\ln\left(\frac{5}{9}\right)$. [5]

$$\int_1^2 (2x-1)^{-2} - \frac{1}{2x-1} + \frac{1}{2x+1} + (2x+1)^{-2}$$

$$\left[\frac{(2x-1)^{-1}}{-1} \times \frac{1}{2} - \frac{1}{2}\ln(2x-1) + \frac{1}{2}\ln(2x+1) + \frac{(2x+1)^{-1}}{-1} \times \frac{1}{2}\right]_1^2$$

$$\left[-\frac{1}{2(2x-1)} - \frac{1}{2}\ln(2x-1) + \frac{1}{2}\ln(2x+1) - \frac{1}{2(2x+1)}\right]_1^2$$

$$\left[\frac{-1}{2(2x-1)} - \frac{1}{2(2x+1)} + \frac{1}{2}\ln\left(\frac{2x+1}{2x-1}\right)\right]_1^2$$

$$\left[\frac{-4}{15} + \frac{1}{2}\ln\left(\frac{5}{3}\right)\right] - \left[-\frac{2}{3} + \frac{1}{2}\ln(3)\right]$$

$$\frac{2}{5} + \frac{1}{2}\ln\left(\frac{5}{9}\right)$$

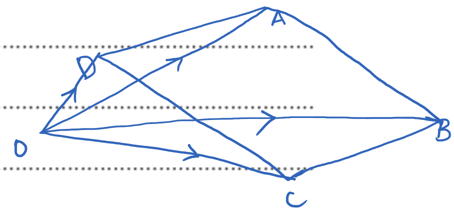
8 Relative to the origin $O$, the points $A$, $B$ and $D$ have position vectors given by

$$\overrightarrow{OA} = \mathbf{i} + 2\mathbf{j} + \mathbf{k}, \quad \overrightarrow{OB} = 2\mathbf{i} + 5\mathbf{j} + 3\mathbf{k} \quad \text{and} \quad \overrightarrow{OD} = 3\mathbf{i} + 2\mathbf{k}.$$

A fourth point $C$ is such that $ABCD$ is a parallelogram.

**(a)** Find the position vector of $C$ and verify that the parallelogram is not a rhombus. [5]

$$\overrightarrow{AB} = \begin{pmatrix}-1\\-2\\-1\end{pmatrix} + \begin{pmatrix}2\\5\\3\end{pmatrix} = \begin{pmatrix}1\\3\\2\end{pmatrix}$$




$$\overrightarrow{AD} = \begin{pmatrix}-1\\-2\\-1\end{pmatrix} + \begin{pmatrix}3\\0\\2\end{pmatrix} = \begin{pmatrix}2\\-2\\1\end{pmatrix}$$

$\overrightarrow{BC}$ is parallel to $\overrightarrow{AD}$ so $\overrightarrow{BC} = \begin{pmatrix}2\\-2\\1\end{pmatrix}$

$$\overrightarrow{OC} = \overrightarrow{OD} + \overrightarrow{DA} + \overrightarrow{AB} + \overrightarrow{BC}$$

$$= \begin{pmatrix}3\\0\\2\end{pmatrix} + \begin{pmatrix}-2\\2\\-1\end{pmatrix} + \begin{pmatrix}1\\3\\2\end{pmatrix} + \begin{pmatrix}2\\-2\\1\end{pmatrix} = \begin{pmatrix}4\\3\\4\end{pmatrix} = 4i + 3j + 4k$$

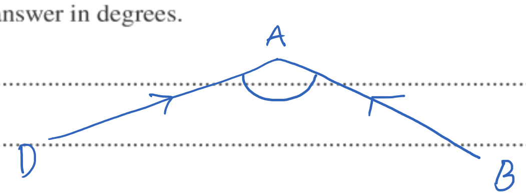
Rhombus will have its diagonals intersectings at 90° but parallelogram won't

$$BD \cdot AC \neq 0$$

$$\begin{pmatrix}1\\-5\\-1\end{pmatrix} \cdot \begin{pmatrix}3\\1\\3\end{pmatrix} = 3 - 5 - 3 = -5$$

**(b)** Find angle *BAD*, giving your answer in degrees. [3]




$\overrightarrow{BA} = \begin{pmatrix} -1 \\ -3 \\ -2 \end{pmatrix}$ & $\overrightarrow{DA} = \begin{pmatrix} -2 \\ 2 \\ -1 \end{pmatrix}$

$\cos\theta \sqrt{(-1)^2+(-3)^2+(-2)^2} \times \sqrt{(-2)^2+(2)^2+(-1)^2} \quad \begin{pmatrix} -1 \\ -3 \\ -2 \end{pmatrix} \cdot \begin{pmatrix} -2 \\ 2 \\ -1 \end{pmatrix}$

$\theta = 100.3^\circ$

**(c)** Find the area of the parallelogram correct to 3 significant figures. [2]

$\left[A \text{ of triangle } BAD\right] \times 2$

$\left[\frac{1}{2} \times \text{length of } AB \times \text{length of } AD \times \sin BAD\right] \times 2$

Length of $AB = \sqrt{(1)^2+(3)^2+(2)^2} = \sqrt{14}$

Length of $AD = \sqrt{(2)^2+(-2)^2+(1)^2} = 3$

$\left[\frac{1}{2} \times \sqrt{14} \times 3 \times \sin(100.3)\right] \times 2 = 11.04$

9 (a) The complex numbers $u$ and $w$ are such that

$$u - w = 2\mathrm{i} \quad \text{and} \quad uw = 6.$$

Find $u$ and $w$, giving your answers in the form $x + \mathrm{i}y$, where $x$ and $y$ are real and exact. [5]

$u = 2i + w$

$(2i + w)w = 6$

$2iw + w^2 = 6$

$w^2 + 2iw - 6 = 0$

$w = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

$w = \dfrac{-2i \pm \sqrt{(2i)^2 - 4(1)(-6)}}{2(1)} = \dfrac{-2i \pm \sqrt{20}}{2}$

$w = -i + \sqrt{5}$ or $-i - \sqrt{5}$

$u = i + \sqrt{5}$ or $i - \sqrt{5}$

**(b)** On a sketch of an Argand diagram, shade the region whose points represent complex numbers $z$ satisfying the inequalities

$$|z - 2 - 2i| \leqslant 2, \quad 0 \leqslant \arg z \leqslant \tfrac{1}{4}\pi \quad \text{and} \quad \text{Re}\, z \leqslant 3.$$

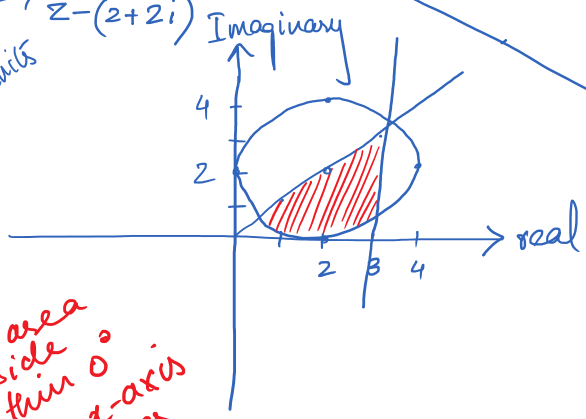
[5]

$z-(2+2i)$

Area under circle with (2, 2) centre & radius of 2 units

area within circle that's less than real value of 3

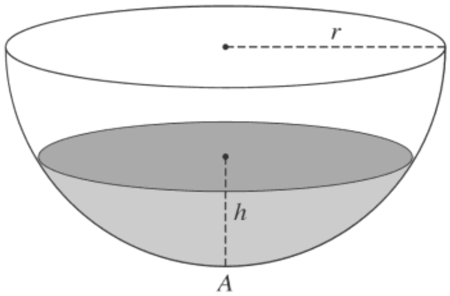
0° to 45° from origin & anticlockwise




We want an area which is inside circle, within 0° to 45° to the x-axis & has real values less than 3.

  **[Turn over**

**10**




$t=0,\ h=r$

A tank containing water is in the form of a hemisphere. The axis is vertical, the lowest point is $A$ and the radius is $r$, as shown in the diagram. The depth of water at time $t$ is $h$. At time $t = 0$ the tank is full and the depth of the water is $r$. At this instant a tap at $A$ is opened and water begins to flow out at a rate proportional to $\sqrt{h}$. The tank becomes empty at time $t = 14$. At $t=14,\ V=0$

$\frac{dV}{dt} \propto -\sqrt{h}$

negative proportionality bc₂ with increase in time, volume is decreasing

The volume of water in the tank is $V$ when the depth is $h$. It is given that $V = \frac{1}{3}\pi(3rh^2 - h^3)$.

**(a)** Show that $h$ and $t$ satisfy a differential equation of the form

$$\frac{dh}{dt} = -\frac{B}{2rh^{\frac{1}{2}} - h^{\frac{3}{2}}},$$

where $B$ is a positive constant. [4]

$V = \pi rh^2 - \frac{\pi h^3}{3}$

$E_1\ \frac{dV}{dt} = -kh^{\frac{1}{2}}$

$\frac{dV}{dh} = 2\pi rh - \pi h^2$

$\frac{dh}{dt} = \frac{dV}{dt} \div \frac{dV}{dh}$

$= \frac{-kh^{\frac{1}{2}}}{2\pi rh - \pi h^2}$

$= \frac{-\cancel{\sqrt{h}}\,k}{\cancel{\sqrt{h}}\left(2\pi r\sqrt{h} - \pi h^{\frac{3}{2}}\right)}$

$\frac{dh}{dt} = \frac{-k}{2\pi rh^{\frac{1}{2}} - \pi h^{\frac{3}{2}}}$

$B = \frac{k}{\pi}$

**(b)** Solve the differential equation and obtain an expression for $t$ in terms of $h$ and $r$. [8]

$\pi$ not included means we'll take $\frac{k}{\pi}$ as B

$$\frac{dh}{dt} = -\frac{B}{2rh^{\frac{1}{2}} - h^{\frac{3}{2}}}$$

Separate h and t

$$\int 2rh^{\frac{1}{2}} - h^{\frac{3}{2}}\, dh = -\int B\, dt$$

$$2\int rh^{\frac{1}{2}} - \int h^{\frac{3}{2}}\, dh = -Bt + C$$

$$2\left[\frac{2}{3}rh^{\frac{3}{2}}\right] - \left[\frac{2}{5}h^{\frac{5}{2}}\right] = -Bt + c$$

$$\frac{4}{3}rh^{\frac{3}{2}} - \frac{2}{5}h^{\frac{5}{2}} = -Bt + c$$

When $t = 0, h = r$ & $t = 14, h = 0$

When $t = 0$, $c = \frac{14}{15}r^{\frac{5}{2}}$ & $B = \frac{r^{\frac{5}{2}}}{15}$

$$\frac{r^{\frac{5}{2}}}{15}t = \frac{14}{15}r^{\frac{5}{2}} - \frac{4}{3}rh^{\frac{3}{2}} + \frac{2}{5}h^{\frac{5}{2}}$$

$$t = \frac{42r^{\frac{5}{2}} - 60rh^{\frac{3}{2}} + 18h^{\frac{5}{2}}}{3r^{\frac{5}{2}}}$$

$$t = 14 - 20\left(\frac{h}{r}\right)^{\frac{3}{2}} + 6\left(\frac{h}{r}\right)^{\frac{5}{2}}$$

## Additional Page

If you use the following lined page to complete the answer(s) to any question(s), the question number(s) must be clearly shown.

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