Cambridge Assessment International Education

Cambridge International AS & A Level

CANDIDATE NAME

CENTRE NUMBER

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* 1 2 1 4 6 4 7 7 5 3 *

MATHEMATICS **9709/31**

Paper 3 Pure Mathematics 3 **May/June 2020**

1 hour 50 minutes

You must answer on the question paper.

You will need: List of formulae (MF19)

INSTRUCTIONS

- Answer **all** questions.
- Use a black or dark blue pen. You may use an HB pencil for any diagrams or graphs.
- Write your name, centre number and candidate number in the boxes at the top of the page.
- Write your answer to each question in the space provided.
- Do **not** use an erasable pen or correction fluid.
- Do **not** write on any bar codes.
- If additional space is needed, you should use the lined page at the end of this booklet; the question number or numbers must be clearly shown.
- You should use a calculator where appropriate.
- You must show all necessary working clearly; no marks will be given for unsupported answers from a calculator.
- Give non-exact numerical answers correct to 3 significant figures, or 1 decimal place for angles in degrees, unless a different level of accuracy is specified in the question.

INFORMATION

- The total mark for this paper is 75.
- The number of marks for each question or part question is shown in brackets [ ].

This document has **20** pages. Blank pages are indicated.

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1 Find the set of values of x for which $2(3^{1-2x}) < 5^x$. Give your answer in a simplified exact form. [4]

$\ln 2(3^{1-2x}) < \ln 5^x$

$\ln 2 + \ln 3^{1-2x} < \ln 5^x$

$\ln 2 + (1-2x)\ln 3 < x\ln 5$

$\ln 2 + \ln 3 - 2x\ln 3 < x\ln 5$

$\ln(2\times 3) < x\ln 5 + 2x\ln 3$

$\ln 6 < x\ln 5 + x\ln 3^2$

$\ln 6 < x(\ln 5 + \ln 9)$

$x > \dfrac{\ln 6}{\ln 5 + \ln 9} \longrightarrow x > \dfrac{\ln 6}{\ln(45)}$

2 **(a)** Expand $(2-3x)^{-2}$ in ascending powers of x, up to and including the term in x^2, simplifying the coefficients. [4]

$(2-3x)^{-2}$ needs to be in the form $(1+ax)^{-2}$

so it becomes $2^{-2}\left(1-\frac{3}{2}x\right)^{-2} = \frac{1}{4}\left(1-\frac{3}{2}x\right)^{-2}$

$b = -\frac{3}{2}$ & $n = -2$

$\frac{1}{4}\left(1 + bn + \frac{n(n-1)}{2!} \times b^2 + \frac{n(n-1)(n-2)}{3!} \times b^3 + \ldots\right)$

$\frac{1}{4}\left(1 + 3x + \frac{27}{4}x^2 + \ldots\right) = \frac{1}{4} + \frac{3}{4}x + \frac{27}{16}x^2$

(b) State the set of values of x for which the expansion is valid. [1]

Expansion of $(1+ax)^n$ is valid for $|ax| < 1$ or $|x| < \frac{1}{a}$

So its either $\left|\frac{3}{2}x\right| < 1$ or $|x| < \frac{2}{3}$

3 Express the equation $\tan(\theta + 60°) = 2 + \tan(60° - \theta)$ as a quadratic equation in $\tan\theta$, and hence solve the equation for $0° \leqslant \theta \leqslant 180°$. [6]

$$\tan(A+B) = \frac{\tan A + \tan B}{1 - \tan A \tan B} \longrightarrow \tan(\theta + 60°) = \frac{\tan\theta + \tan 60}{1 - \tan 60 \tan\theta} = \frac{\tan\theta + \sqrt{3}}{1 - \sqrt{3}\tan\theta}$$

$$\tan(A-B) = \frac{\tan A - \tan B}{1 + \tan A \tan B} \longrightarrow \tan(60° - \theta) = \frac{\tan 60° - \tan\theta}{1 + \tan 60° \tan\theta} = \frac{\sqrt{3} - \tan\theta}{1 + \sqrt{3}\tan\theta}$$

$$\frac{\tan\theta + \sqrt{3}}{1 - \sqrt{3}\tan\theta} = 2 + \frac{\sqrt{3} - \tan\theta}{1 + \sqrt{3}\tan\theta}$$

$$\frac{\tan\theta + \sqrt{3}}{1 - \sqrt{3}\tan\theta} - \frac{(\sqrt{3} - \tan\theta)}{1 + \sqrt{3}\tan\theta} = 2$$

$$\frac{(\tan\theta + \sqrt{3})(1 + \sqrt{3}\tan\theta) - (\sqrt{3} - \tan\theta)(1 - \sqrt{3}\tan\theta)}{1 - 3\tan^2\theta} = 2$$

$$\frac{\tan\theta + \sqrt{3}\tan^2\theta + \sqrt{3} + 3\tan\theta - (\sqrt{3} - 3\tan\theta - \tan\theta + \sqrt{3}\tan^2\theta)}{1 - 3\tan^2\theta} = 2$$

$$\frac{4\tan\theta + \cancel{\sqrt{3}} + \cancel{\sqrt{3}\tan^2\theta} - \cancel{\sqrt{3}} + 3\tan\theta + \tan\theta - \cancel{\sqrt{3}\tan^2\theta}}{1 - 3\tan^2\theta} = 2$$

$$8\tan\theta = 2 - 6\tan^2\theta$$

$$6\tan^2\theta + 8\tan\theta - 2 = 0$$

$$3\tan^2\theta + 4\tan\theta - 1 = 0$$

$$\tan\theta = \frac{-2 + \sqrt{7}}{3}, \quad \frac{-2 - \sqrt{7}}{3}$$

$$\theta = 12 \cdot 1476°, \quad \theta = -57 \cdot 1476°$$

Tan θ is also +ve in 1st quadrant where its just θ so its $12 \cdot 15°$

$\theta = 12 \cdot 15°$

Tan θ is +ve in 3rd quadrant where its $180 + \theta°$ so its $180 + 12 \cdot 1476$

$\theta = 192 \cdot 1476°$

Since this isnt in range of $0 \leq \theta \leq 180°$

Tan θ is −ve in 2nd quadrant where its $180 - \theta$ so its $180 - 57 \cdot 1476$

negative sign can be ignored here of 57.1476

$\theta = 122 \cdot 85$

4 The curve with equation $y = e^{2x}(\sin x + 3\cos x)$ has a stationary point in the interval $0 \leqslant x \leqslant \pi$.

(a) Find the x-coordinate of this point, giving your answer correct to 2 decimal places. [4]

To find stationary point, $\frac{dy}{dx} = 0$

$y = e^{2x}\sin x + 3e^{2x}\cos x$

How to differentiate in which 2 expressions are together? You must differentiate one at a time

Break $e^{2x}\sin x$ to $u = e^{2x}$ & $v = \sin x$

$\frac{du}{dx} = 2e^{2x}$ $\frac{dv}{dx} = \cos x$

Similarly break $3e^{2x}\cos x$ to $u = 3e^{2x}$ & $v = \cos x$

$\frac{du}{dx} = 6e^{2x}$ $\frac{dv}{dx} = -\sin x$

Now always follow this method.

$\left[u\frac{dv}{dx} + v\frac{du}{dx} \text{ of } e^{2x}\sin x\right] + \left[u\frac{dv}{dx} + v\frac{du}{dx} \text{ of } 3e^{2x}\cos x\right]$

$\left[e^{2x} \times \cos x + \sin x(2e^{2x})\right] + \left[3e^{2x}(-\sin x) + \cos x(6e^{2x})\right]$

$\frac{dy}{dx} \rightarrow e^{2x}\cos x + 2e^{2x}\sin x - 3e^{2x}\sin x + 6e^{2x}\cos x = 0$

$7e^{2x}\cos x - e^{2x}\sin x = 0 \longrightarrow \tan x = 7 \longrightarrow x = 1.43$ radians

(b) Determine whether the stationary point is a maximum or a minimum. [2]

Now for this, use $\frac{d^2y}{dx^2}$

$\frac{dy}{dx} = 7e^{2x}\cos x - e^{2x}\sin x$. Differentiate this in the same manner as we did in part (a)

$\frac{d^2y}{dx^2} = 13e^{2x}\cos x - 9e^{2x}\sin x$

Now substitute $x = 1.43$ in $\frac{d^2y}{dx^2}$

$\frac{d^2y}{dx^2} = -123.7 < 0$ hence it's a maximum point

5 **(a)** Find the quotient and remainder when $2x^3 - x^2 + 6x + 3$ is divided by $x^2 + 3$. [3]

$$\frac{2x^3 - x^2 + 6x + 3}{x^2 + 3}$$

$2x - 1$ ← quotient

$$x^2+3 \,\big)\, 2x^3 - x^2 + 6x + 3$$

$$-(2x^3 + 0x^2 + 6x) \downarrow$$

$$-x^2 + 3$$

$$-(-x^2 - 3)$$

$0 + 6$ → remainder

① Multiply $2x$ with x^2+3 which gives $-x^2+3$ in remainder.

② Multiply -1 with x^2+3 that gives $+6$ as remainder.

(b) Using your answer to part **(a)**, find the exact value of $\displaystyle\int_1^3 \frac{2x^3 - x^2 + 6x + 3}{x^2 + 3}\,dx$. [5]

$\dfrac{2x^3 - x^2 + 6x + 3}{x^2+3}$ had $2x-1$ as quotient and 6 as remainder

so it can be written as $2x - 1 + \dfrac{6}{x^2+3}$

$$\int_1^3 2x - 1 + \frac{6}{x^2+3}\,dx$$

$$\Big[x^2 - x\Big]_1^3 + 6\int \frac{1}{x^2+3}\,dx$$

Suppose $x = \sqrt{3}\tan\theta$ then $x^2 = 3\tan^2\theta$

& $\dfrac{dx}{d\theta} = \sqrt{3}\sec^2\theta$

$$\Big[x^2 - x\Big]_1^3 + 6\int_1^3 \frac{\sqrt{3}\sec^2\theta}{3(\tan^2\theta + 1)}\,d\theta$$

$d\theta \rightarrow$ equal to dx

$(\tan^2\theta + 1) \rightarrow \sec^2\theta$

$$\Big[x^2 - x\Big]_1^3 + 6\int_1^3 \frac{\sqrt{3}\,\cancel{\sec^2\theta}}{3\,\cancel{\sec^2\theta}}\,d\theta$$

$$\Big[x^2 - x\Big]_1^3 + 6\int_1^3 \frac{1}{\sqrt{3}}\,d\theta$$

$$\Big[x^2 - x\Big]_1^3 + 6\left[\frac{\theta}{\sqrt{3}}\right]$$

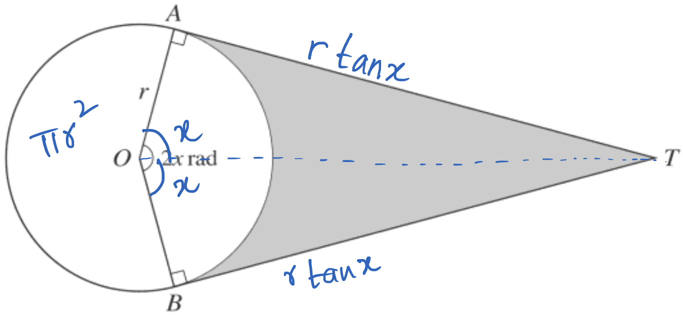
Now we must bring x back in θ's position

θ was equal to $\tan^{-1}\left(\dfrac{x}{\sqrt{3}}\right)$

$$\left[x^2 - x + 2\sqrt{3}\tan^{-1}\left(\frac{x}{\sqrt{3}}\right)\right]_1^3$$

$$6 + 2\sqrt{3}\tan^{-1}(\sqrt{3}) - 2\sqrt{3}\tan^{-1}\left(\frac{1}{\sqrt{3}}\right) = \frac{\pi}{\sqrt{3}} + 6$$

6



The diagram shows a circle with centre O and radius r. The tangents to the circle at the points A and B meet at T, and angle AOB is $2x$ radians. The shaded region is bounded by the tangents AT and BT, and by the minor arc AB. The area of the shaded region is equal to the area of the circle.

(a) Show that x satisfies the equation $\tan x = \pi + x$. [3]

Area of shaded region = area of circle

Area of shaded region = (area of OAT) × 2 − A of sector OAB

$= \left(\frac{1}{\cancel{2}} \times r \times r\tan x\right) \times \cancel{2} - \left(\frac{1}{\cancel{2}} r^2 \times \cancel{2}x\right)$

A of shaded region $= r^2 \tan x - r^2 x$

A of circle $= \pi r^2$

$\cancel{r^2}(\tan x - x) = \pi \cancel{r^2}$

$\tan x = \pi + x$ //

(b) This equation has one root in the interval $0 < x < \frac{1}{2}\pi$. Verify by calculation that this root lies between 1 and 1.4. [2]

$\tan x - x - \pi$

When $x = 1$, its -2.584

when $x = 1.4$, its 1.2563

When 1 is negative & the other is positive, it means it was zero somewhere in between

(c) Use the iterative formula

$$x_{n+1} = \tan^{-1}(\pi + x_n)$$

to determine the root correct to 2 decimal places. Give the result of each iteration to 4 decimal places. [3]

$x_{1+1} = \tan^{-1}(\pi + 1) = 1.3339 \longrightarrow x_2 = 1.3339$

$x_{2+1} = \tan^{-1}(\pi + 1.334) = 1.3510 \longrightarrow x_3 = 1.3510$

$x_{3+1} = \tan^{-1}(\pi + 1.351) = 1.3518 \longrightarrow x_4 = 1.3518$

$x_{4+1} = \tan^{-1}(\pi + 1.355) = 1.352 \longrightarrow x_5 = 1.3520$

$\therefore x = 1.35$

7 Let $f(x) = \dfrac{\cos x}{1 + \sin x}$.

(a) Show that $f'(x) < 0$ for all x in the interval $-\frac{1}{2}\pi < x < \frac{3}{2}\pi$. [4]

$$y = \frac{\cos x}{1+\sin x}$$

Make $\cos x = u$ & $1+\sin x = v$ & follow this method

$$\left(v\frac{du}{dx} - u\frac{dv}{dx}\right) \div v^2$$

$u = \cos x$ so $\dfrac{du}{dx} = -\sin x$ $\quad$ $v = 1+\sin x$ so $\dfrac{dv}{dx} = \cos x$

$$f'(x) = \frac{(1+\sin x)(-\sin x) - \cos x(\cos x)}{(1+\sin x)^2}$$

equal to 1

$$\frac{-\sin x - \sin^2 x - \cos^2 x}{(1+\sin x)^2} = \frac{-\sin x - 1(\sin^2 x + \cos^2 x)}{(1+\sin x)^2}$$

$$\frac{-\sin x - 1}{(1+\sin x)^2} < 0 \text{ when } \frac{\pi}{2} < x < \frac{3\pi}{2}$$

(b) Find $\int_{\frac{1}{6}\pi}^{\frac{1}{2}\pi} f(x)\,dx$. Give your answer in a simplified exact form. [4]

$$\int \frac{\cos x}{1+\sin x}\,dx$$

Assume $u = 1+\sin x$

$$\frac{du}{dx} = \cos x \quad \text{then} \quad \cos x\,dx = du$$

$$\int \frac{\cos x}{1+\sin x}\,dx \longrightarrow \int \frac{1}{u}\,du$$

$$\Big[\ln u\Big]_{\frac{\pi}{6}}^{\frac{\pi}{2}}$$

Since $u = 1+\sin x \longrightarrow \Big[\ln|1+\sin x|\Big]_{\frac{\pi}{6}}^{\frac{\pi}{2}}$

$$\ln 2 - \ln\frac{3}{2} = \ln\frac{4}{3}$$

8 A certain curve is such that its gradient at a point (x, y) is proportional to $\frac{y}{x\sqrt{x}}$. The curve passes through the points with coordinates $(1, 1)$ and $(4, \mathrm{e})$.

(a) By setting up and solving a differential equation, find the equation of the curve, expressing y in terms of x. [8]

A curve's gradient is $\frac{dy}{dx}$ so $\frac{dy}{dx} \propto \frac{y}{x\sqrt{x}}$

$\frac{dy}{dx} = \frac{ky}{x\sqrt{x}}$ Separate x & y on both sides & integrate both sides

$$\int \frac{dy}{y} = \int \frac{k}{x\sqrt{x}}\,dx$$

$$\ln y = k\int \frac{1}{x^{3/2}}\,dx \longrightarrow \ln y = k\int x^{-3/2}\,dx$$

$$\ln y = k\left[\frac{x^{-\frac{3}{2}+1}}{-\frac{3}{2}+1} + c\right] \longrightarrow \ln y = -2kx^{-\frac{1}{2}} + kc$$

Now we can use $(1,1)$ & $(4,e)$ to find values for c and k.

$$\ln(1) = -2k(1)^{-\frac{1}{2}} + kc \longrightarrow c = 2$$

$$\ln e = -2k(4)^{-\frac{1}{2}} + 2k \longrightarrow k = 1$$

$$\ln y = -\frac{2}{\sqrt{x}} + 2$$

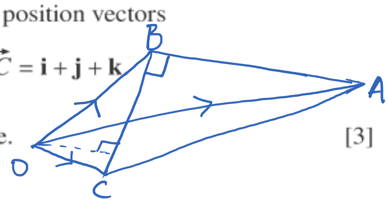
$$y = e^{2 - \frac{2}{\sqrt{x}}}$$

(b) Describe what happens to y as x tends to infinity. [1]

$\frac{-2}{\sqrt{x}}$ gets nearer to zero so y approaches e^2

9 With respect to the origin O, the vertices of a triangle ABC have position vectors

$$\overrightarrow{OA} = 2\mathbf{i} + 5\mathbf{k}, \quad \overrightarrow{OB} = 3\mathbf{i} + 2\mathbf{j} + 3\mathbf{k} \quad \text{and} \quad \overrightarrow{OC} = \mathbf{i} + \mathbf{j} + \mathbf{k}.$$



(a) Using a scalar product, show that angle ABC is a right angle. [3]

$\overrightarrow{AB} = \overrightarrow{AO} + \overrightarrow{OB}$

$= \begin{pmatrix} -2 \\ 0 \\ -5 \end{pmatrix} + \begin{pmatrix} 3 \\ 2 \\ 3 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ -2 \end{pmatrix}$

$\overrightarrow{CB} = \overrightarrow{CO} + \overrightarrow{OB}$

$= \begin{pmatrix} -1 \\ -1 \\ -1 \end{pmatrix} + \begin{pmatrix} 3 \\ 2 \\ 3 \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \\ 2 \end{pmatrix}$

$\begin{pmatrix} 1 \\ 2 \\ -2 \end{pmatrix} \cdot \begin{pmatrix} 2 \\ 1 \\ 2 \end{pmatrix} = 2 + 2 - 4 = 0$

Scalar product is zero thus ABC is a right angle triangle

(b) Show that triangle ABC is isosceles. [2]

Length of AB = length of BC

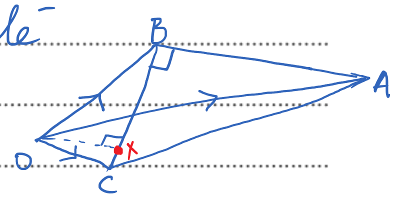
$\sqrt{(1)^2+(2)^2+(-2)^2} = \sqrt{(-2)^2+(-1)^2+(-2)^2}$

$3 = 3$

Triangle ABC is isosceles

(c) Find the exact length of the perpendicular from O to the line through B and C. [4]

For this we should first calculate vector equation of line BC



$$r = \underbrace{c}_{\text{position vector of B}} + \lambda \underbrace{b}_{\text{vector of BC}}$$

$$r = 3i + 2j + 3k + \lambda(-2i - j - 2k) \rightarrow \begin{pmatrix} 3-2\lambda \\ 2-\lambda \\ 3-2\lambda \end{pmatrix}$$

hence $\overrightarrow{OX}$ is $\begin{pmatrix} 3-2\lambda \\ 2-\lambda \\ 3-2\lambda \end{pmatrix}$ bcz X is a general point on BC

$\overrightarrow{OX}$ is perpendicular to $\overrightarrow{BC}$

$$\begin{pmatrix} 3-2\lambda \\ 2-\lambda \\ 3-2\lambda \end{pmatrix} \cdot \begin{pmatrix} -2 \\ -1 \\ -2 \end{pmatrix} = 0$$

$$-6 + 4\lambda - 2 + \lambda - 6 + 4\lambda = 0 \quad \text{so } \lambda = \frac{14}{9}$$

$$\overrightarrow{OX} = \begin{pmatrix} 3 - 2(14/9) \\ 2 - 14/9 \\ 3 - 2(14/9) \end{pmatrix} = -\frac{1}{9}i + \frac{4}{9}j - \frac{1}{9}k$$

Length of $\overrightarrow{OX} = \dfrac{\sqrt{2}}{3}$

[Turn over

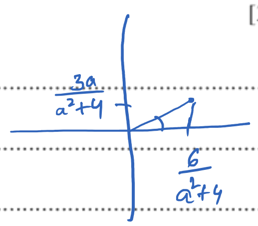
10 **(a)** The complex number u is defined by $u = \dfrac{3\mathrm{i}}{a + 2\mathrm{i}}$, where a is real.

(i) Express u in the Cartesian form $x + \mathrm{i}y$, where x and y are in terms of a. [3]

$$\frac{3i}{(a+2i)} \times \frac{(a-2i)}{(a-2i)} = \frac{3ia - 6i^2}{a^2 - 4i^2} \qquad i^2 = -1$$

$$\frac{3ai + 6}{a^2 + 4} \longrightarrow \frac{6}{a^2+4} + \frac{3ai}{a^2+4}$$

(ii) Find the exact value of a for which $\arg u^* = \frac{1}{3}\pi$. [3]



$$\arg(u) = -\frac{\pi}{3}$$

$$\tan\left(-\frac{\pi}{3}\right) = \frac{\cancel{3}a}{\cancel{a^2+4}} \times \frac{\cancel{a^2+4}}{\cancel{6}\,2}$$

$$-\sqrt{3} = \frac{a}{2}$$

$$a = -2\sqrt{3}$$

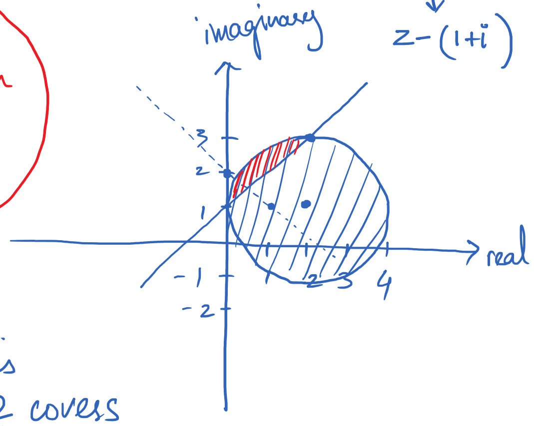
$u = \dfrac{3i}{\quad}$

$$u = \frac{3i}{-2\sqrt{3}+2i}$$

(b) **(i)** On a sketch of an Argand diagram, shade the region whose points represent complex numbers z satisfying the inequalities $|z - 2i| \leqslant |z - 1 - i|$ and $|z - 2 - i| \leqslant 2$. [4]

$z-(1+i)$ $\quad$ $z-(2+i)$

$|z-2i| \leq |z-(1+i)|$ means a line which is a perpendicular bisector of a line that joins (0,2) & (1,1)

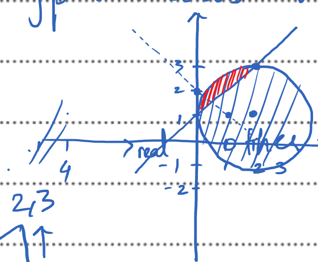


$|z-(2+i)| \leqslant 2$ means a circle with 2+i as centrepoint & radius of 2 units

Blue shaded part is what $|z-(2+i)| \leq 2$ covers & red one is for this Q where shaded region is going to be part of circle thats nearer to (0,2)

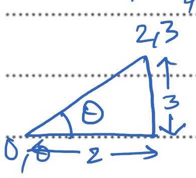
(ii) Calculate the least value of arg z for points in this region. [2]

Least arg z is least angle from origin to the hypoteneous that you've to decide on



If you take any point other than (2,3) & that being from shaded region will give you larger angle from +ve x-axis

Hence I must take a point from locus available & one which is nearer to +ve x axis & thats (2,3)



$\tan\theta = \frac{3}{2}$ so $\theta = 56.3°$

Additional Page

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